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How Cold Dark Matter Theory Explains Milgrom's Law

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ABSTRACT

Milgrom noticed the remarkable fact that the gravitational effect of dark matter in galaxies only becomes important where accelerations are less than about $10^{-8} \text{ cm/s}^2 \sim cH_0$ (“Milgrom’s Law”). This forms the basis for Modified Newtonian Dynamics (MOND), an alternative to particle dark matter. However, any successful theory of galactic dynamics must account for Milgrom’s Law. We show how Milgrom’s Law comes about in the Cold Dark Matter (CDM) theory of structure formation.

Subject headings: cosmology: dark matter—galaxies: dynamics

1. Introduction

The dark-matter mystery has been with us since Zwicky noticed that the gravitational action of luminous matter is not sufficient to hold clusters together (Zwicky 1933; Smith 1936). Rubin and others brought the problem closer to home by showing that spiral galaxies like ours suffer the same problem (see e.g., Knapp and Kormendy (1987)). While the leading explanation for the dark matter problem today is slowly moving, weakly interacting “nonluminous” elementary particles remaining from the earliest moments – cold dark matter (see e.g., Turner (2000)) – there is still interest in the possibility that the explanation involves new gravitational physics (see e.g., Sellwood and Kosowsky (2000)). It is important to realize that particle dark matter does exist – the SuperKamiokande evidence (Fukuda et al 1998)

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for neutrino mass implies that neutrino dark matter accounts for as much, or perhaps more, matter as do bright stars.

Any gravitational explanation must deal with the fact that the shortfall of the Newtonian gravity of luminous matter occurs at widely different length scales – at distances much less than 1 kpc in dwarf spirals to distances greater than 100 kpc in clusters of galaxies. Merely strengthening gravity beyond a fixed distance cannot explain away the need for dark matter.

In 1983 Milgrom (1983a,b) made a remarkable observation: the need for the gravitational effect of nonluminous (dark) matter in galaxies only arises when the Newtonian acceleration is less than about $a_0 = 2 \times 10^{-8} \text{ cm s}^{-2} = 0.3 cH_0$. (Here, $H_0 = 70 \pm 7 \text{ km s}^{-1} \text{ Mpc}^{-1} = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$ is present expansion rate of the Universe.) This fact, which we will refer to as Milgrom’s law, is the foundation for his Modified Newtonian Dynamics (MOND) alternative to particle dark matter. It is not our claim that the analysis to follow rules out MOND.

The correctness or incorrectness of MOND aside, the empirical fact that the need for dark matter in galaxies always seems to occur at an acceleration of around cH_0 must be explained by a successful theory of structure formation. This *Letter* shows how Milgrom’s Law arises in the cold dark matter theory of structure formation.

2. How CDM Predicts Milgrom’s Law

2.1. CDM theory

The cold dark matter theory of structure formation has two basic features: seed density inhomogeneity that arose from quantum fluctuations during inflation, and dark matter existing in the form of slowly moving particles left over from the big bang. The two leading candidates for the CDM particle are the axion and the neutralino. Each is predicted by a compelling extension of the standard model of particle physics motivated by particle-physics considerations (rather than cosmological) and has a predicted relic density comparable to that of the known matter density (see e.g., Turner (2000)).

A recent estimate of the matter density puts the total at $\Omega_M = 0.330 \pm 0.035$ and baryons at $\Omega_B = 0.040 \pm 0.008$ (Turner 2001). This means that CDM particles contribute $\Omega_{\text{CDM}} = 0.29 \pm 0.04$ (less the contribution of neutrinos). (Croft et al (1999) argue based upon the formation of small-scale structure, that neutrinos can contribute no more than about 10% of the critical density.)

For our purposes here, a less essential feature of CDM is the fact that the bulk of the

critical density exists in the form of a mysterious dark energy ($\Omega_X \simeq 0.66 \pm 0.06$; see e.g., Turner (2001)). While the existence of dark energy affects the details of structure formation enough so that observations can discriminate between a matter-dominated flat Universe and one with dark energy, for the purposes of showing how CDM predicts Milgrom’s Law, dark energy and its character are not critical. This is because most galaxies formed while the Universe was still matter-dominated and well described by the Einstein – deSitter model.

In the CDM scenario, structure forms from the bottom up, through hierarchical merging of small halos to form larger halos (see e.g., Blumenthal et al (1984)). The bulk of galactic halos formed around redshifts of 1 to 5, with clusters forming at redshifts of 1 or less, and superclusters forming today. Within halos, baryons lose energy through electromagnetic interactions and sink to the center, supported by their angular momentum. Until baryonic dissipation occurs, baryons and CDM particles exist in a universal ratio of $\Omega_{\text{CDM}}/\Omega_B \simeq 7$. Were it not for the concentration of baryons caused by dissipation, the gravity of dark matter would be dominant everywhere.

2.2. CDM and Milgrom’s Law

The CDM explanation for the gravitational effect of dark matter “kicking in” at a fixed acceleration approximately equal to cH_0 involves three ingredients: i) the fact that the Universe is reasonably well described by the Einstein – deSitter model during the period when galaxies form; ii) the scale-free character of the seed density perturbations over the relevant scales; iii) baryonic dissipation; and iv) numerical coincidences.

The argument begins with facts i) and ii), which lead to the CDM prediction of self-similar dark-matter halos. Halos, regardless of their mass, can be described by the same mathematical form (Navarro et al 1997, henceforth NFW). The exact functional form is not essential (see below); for simplicity we write the halo profile for an object that began from perturbations of comoving length scale L as

$$\rho_L(r) \simeq \beta^3 \Omega_M \rho_{\text{crit}} (1 + z_c)^3 (r/\ell)^{-2} = \beta \Omega_M \rho_{\text{crit}} (1 + z_c) (r/L)^{-2}, \quad (1)$$

where $\rho_{\text{crit}} = 3H_0^2/8\pi G$ is the critical density today, z_c is the redshift of halo collapse and β is a numerical constant of $\mathcal{O}(5)$. The physical size of the perturbation after collapse ($\equiv \ell$) is related to its comoving size, $\ell = L/\beta(1 + z_c)$; the factor of $1/(1 + z_c)$ is due to the expansion of the Universe and the factor of $1/\beta$ is due to collisionless collapse. Because $\Omega_M(1 + z_c)^3 \rho_{\text{crit}}$ is the mean matter density at the redshift of collapse, Eq. 1 says that the mean density of the collapsed structure interior to $r = \ell$ is about 100 times the ambient density when collapse occurred.

The redshift of collapse is determined by the spectrum of density perturbations: collapse on length scale L occurs when the *rms* mass fluctuation on that scale ($\equiv \sigma_L$) is of order unity. Neglecting nonlinear effects, σ_L at redshift z is related to the matter power spectrum today ($\equiv |\delta_k|^2$):

$$\sigma_L(z) = \left[\int_0^\infty \frac{k^2 |\delta_k|^2}{2\pi^2} |W_L(k)|^2 dk \right]^{1/2} \simeq (\epsilon/10^{-5}) (1+z)^{-1} (L/L_0)^{-\frac{1}{2}(n_{\text{eff}}+3)}, \quad (2)$$

where $k \sim L^{-1}$, $W_L(k)$ is the Fourier transform of the top-hat window function, and $n_{\text{eff}} \approx -2.2$ is the logarithmic slope of $L^3 \sigma_L^2 \sim |\delta_k|^2$ (with respect to k) around galaxy scales.⁴ The quantity ϵ is the dimensionless amplitude of the primeval fluctuations in the gravitational potential, determined by COBE to be about 10^{-5} , and $L_0 \simeq 10h^{-1} \text{ Mpc}$ is the scale of nonlinearity today (for $\epsilon \sim 10^{-5}$). Substituting Eq. 2 into Eq. 1, it follows that

$$\rho_L(r) = [(3\beta/8\pi) \Omega_M (\epsilon/10^{-5})] (H_0^2/G) (L/L_0)^{-\frac{1}{2}(n_{\text{eff}}+3)} (r/L)^{-2}. \quad (3)$$

The third ingredient is baryonic dissipation: after halos form, their baryons dissipate energy and collapse in linear scale by a factor $\alpha \approx 10$ to form a disk supported by angular momentum (see e.g., Dalcanton et al (1997)). The degree of baryonic collapse is determined by the dimensionless spin parameter λ , which is the ratio of the angular velocity of the galaxy to the angular velocity that would be required to support the structure purely by rotation. The angular momentum of galaxies is thought to arise from tidal torquing (Peebles 1969, and references therein). Theory and simulations (Warren et al 1992) seem to agree that λ is independent of scale, with a median value of $\lambda \approx 0.05$. If one assumes that the angular momentum of the gas is conserved during disk formation, then (see Padmanabhan (1993)) $\alpha \approx \Omega_B \lambda_D^2 / \Omega_M \lambda^2$, which is about 12 because the disk spin parameter $\lambda_D \approx 0.5$.

Because of the increased concentration of baryons interior to $r \sim \ell (\Omega_B / \Omega_{\text{CDM}})$, their gravity will dominate the dynamics in the inner regions. (This statement is true as long as $\alpha > \Omega_{\text{CDM}} / \Omega_B$.) Thus, the transition from dark-matter dominated gravity to luminous-matter dominated gravity should occur around $r_{\text{DM}} = L/7\beta(1+z_c)$. The acceleration at the point when dark matter gravity begins to dominate is

$$a_{\text{DM}} \equiv a(r_{\text{DM}}) = \frac{GM(r_{\text{DM}})}{r_{\text{DM}}^2} = [4\pi G (\ell/7)] \rho_L(\ell/7).$$

⁴For exactly scale-invariant density perturbations, n_{eff} varies from -2.5 to -2 for $L = 0.01 \text{ Mpc}$ to $L = 1 \text{ Mpc}$ in SCDM. Inflation does not predict precisely scale-invariant density perturbations (see e.g., Huterer and Turner (2000)). In the case of nonscale-invariant density perturbations, $n_{\text{eff}} = -2.2 + (n - 1)$, where $n - 1$ quantifies the deviation from scale invariance and is expected to be of order ± 0.1 .

After some re-writing, Milgrom’s Law emerges

$$\begin{aligned} a_{\text{DM}} &= cH_0 \left[10\beta^2 \Omega_M \left(\frac{\epsilon}{10^{-5}} \right)^2 \right] (c^{-1}H_0 L_0) \left(\frac{L}{L_0} \right)^{-n_{\text{eff}} - 2}, \\ &= \mathcal{O}(1) cH_0 \left(\frac{L}{L_0} \right)^{0.2}. \end{aligned} \quad (4)$$

The final ingredient is the conspiracy of numerical factors to give a coefficient of unity and a very mild scale dependence (over 3 orders of magnitude in mass, a_{DM} changes by only a factor of 1.6).

We have assumed in the above discussion that most of the baryons in the protogalaxy dissipate and form disks. How valid is this assumption? Clearly some of the baryons will be inhibited from collapsing by the UV radiation field, or blown away into the inter-galactic medium due to feedback from supernovae and possibly other phenomena related to star formation. It is sensible to assume that these effects are more pronounced for smaller mass galaxies. However, so long as the fraction of baryonic matter that collapses does not vary strongly with scale our analysis goes through with only numerical factors changing. In fact, if the collapsed fraction in a $0.1\mathcal{L}_\star$ galaxy (\mathcal{L} denotes the luminosity) is about half that of a \mathcal{L}_\star galaxy, this would give rise to a scale dependence in a_{DM} about the same as, but *opposing* the change, in Equation 4 – leaving a_{DM} essentially scale-free. Of course, at the low mass end of the galactic scale, one could have a much smaller collapsed fraction that could introduce scatter or deviation in the a_{DM} vs luminosity relation (even after taking into account the fact that on those small scales n_{eff} is smaller than -2.2). In this case, an accurate derivation of Milgrom’s law will require more sophisticated models incorporating gas dynamics of the baryons.

The mild scale dependence of the acceleration where dark matter dominates owes to the fact that $n_{\text{eff}} \approx -2$, around galactic scales. It arises from a combination of the primeval spectral index ($n \simeq 1$) and the bending of the shape of the spectrum of perturbations caused by the fact that perturbations on small scales ($k \gtrsim 0.1 \text{ Mpc}^{-1}$) entered the horizon when the Universe was radiation-dominated and those on large scales ($k \lesssim 0.1 \text{ Mpc}^{-1}$) entered the horizon when the Universe was matter-dominated. For $k \ll 0.1 \text{ Mpc}^{-1}$, $n_{\text{eff}} \rightarrow 1$ and for $k \gg 0.1 \text{ Mpc}^{-1}$, $n_{\text{eff}} \rightarrow -3$.

Returning to the numerical conspiracy that leads to $a_{\text{DM}} \approx cH_0$; for $n_{\text{eff}} = -2$, the factor $(\epsilon/10^{-5})^2 L_0$ is just the scale of nonlinearity today, independent of the actual value of ϵ . The numerical coincidence then is the fact that the scale of nonlinearity today is much less than the Hubble scale. Scott et al (2001) have tied this fact to the cooling scale of baryons, which can be related to fundamental constants and ϵ .

Equation 4 only holds around galaxy scales ($L \sim 1$ Mpc), where $n_{\text{eff}} \approx -2$ and $\alpha \sim 10$. Clusters are dark-matter dominated almost everywhere because cluster baryons do not dissipate significantly. Milgrom’s law would, therefore, assert that the Newtonian acceleration in clusters should be less than cH_0 almost everywhere – in contradiction with observations. Said another way, CDM correctly predicts that Milgrom’s Law *should not* apply to clusters.

The issue of the shape of the halo density profile is not central to our arguments. We have repeated our calculation for the NFW profile and find $a_{\text{DM}} \approx 10^{-8} (M_{200}/10^{12} M_{\odot})^{0.1} \text{ cm s}^{-2}$, which is similar to the result obtained in Eq. 4, (M_{200} is the mass interior to the point where the density is 200 times the critical density). MOND automatically predicts asymptotically flat rotation curves; in CDM the flatness of rotation curves has its origin in the fact that over a significant portion of the halo, $\rho_{\text{halo}} \propto r^{-2}$. The NFW halo profile asymptotes to $\rho_{\text{halo}} \rightarrow r^{-3}$ so that CDM predicts $v_{\text{circular}} \rightarrow \sqrt{\ln r/r}$.

Another coincidence for CDM is known. The galaxy-galaxy correlation function is very well fit by a power law, $\xi(r) = (r/r_0)^{-1.8}$ where $r_0 = 5h^{-1}$ Mpc (see e.g., Groth and Peebles (1977); Baugh (1996)). In CDM theory, the two-point correlation function of mass is not a good power law; however, when bias is taken into account (the nontrivial relation between mass and light), the galaxy – galaxy correlation function turns into a power-law (see e.g., Pearce et al (1999)), in good agreement with observations.

3. Concluding remarks

The derivation of Eq. 4 is the key result of this paper. It illustrates how Milgrom’s Law – the need for dark matter in galaxies at accelerations less than about cH_0 – arises in CDM theory. While scale-free density perturbations, an epoch where the Universe is well described by the Einstein – deSitter model and baryonic dissipation are essential, the fact that a_{DM} is nearly cH_0 appears to be a numerical coincidence. Furthermore, a_{DM} is a fixed number since galaxies are bound and well relaxed today, while cH decreases with time. Thus, the approximate equality of a_{DM} with cH only holds today.

The purpose of our *Letter* was to illuminate the basic physics that underlies the emergence of Milgrom’s Law within CDM theory. It was not our intention to present a detailed analysis. To achieve our purpose we made some strong – but we believe reasonable – assumptions. The strongest of these assumptions is that all the baryonic matter associated with galaxies dissipates and collapses. This is probably not true, as significant amounts of baryonic material still exist in hot gas (Fukugita et al 1998). However, so long as the fraction of baryonic matter that collapses does not vary much with scale (which we have quantified

in the previous section) our analysis goes through with only numerical factors changing. If the fraction of baryonic matter that collapses does vary dramatically with scale, one would expect deviant objects and scatter. We remind the reader that a detailed, semi-analytic calculation of galactic rotation curves in CDM (van den Bosch and Dalcanton 2000) clearly shows the presence of a characteristic acceleration scale (of the order of cH_0), and they fit the data about as well as those derived from MOND. (We would argue that Eq. 4 is the underlying explanation for the appearance of this acceleration scale.) Further, a study of about 1000 spiral galaxies (Persic et al 1996), with luminosities from about $0.1\mathcal{L}_*$ to \mathcal{L}_* , is in agreement with r_{DM}/ℓ being approximately constant.⁵ We have verified that for these galaxies the variation in a_{DM} over the 1 order of magnitude range in luminosity is less than 20%. Both of these studies lend credence to our underlying assumptions.

Separating the important clues from the misleading coincidences is at the heart of scientific creativity. Hoyle’s observation that the energy released in burning 25% of the Hydrogen to Helium is approximately equal to that of the CMB suggested a non big-bang origin for the CMB (see e.g., Burbidge & Hoyle, 1998). In the end, it turned out to be a misleading coincidence. Within the big-bang model, Hoyle’s coincidence is explained by the near equality of the dimensionless amplitude of density perturbations ϵ and the product of the efficiency of nuclear burning times Ω_B : To make stars by the present epoch, Ω_γ must be $\sim \epsilon$, which coincidentally is equal to the energy that would be released in producing the observed Helium abundance (Martin Rees, private communication).

CDM not only predicts Milgrom’s Law (at least over a order of magnitude range in luminosity from $0.1\mathcal{L}_*$ to \mathcal{L}_*), but also accounts for a wealth of other cosmological observations. This suggests to us that Milgrom’s Law is a misleading coincidence rather than evidence for a modification of Newtonian dynamics.

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⁵Persic et al (1996) find a relationship between two observationally based quantities: R_T , which is essentially scales as our r_{DM} , and \mathcal{L} . Assuming r_{DM}/ℓ is constant and the observationally inferred baryonic mass vs. luminosity relation (Salucci and Persic 1999), we are able to reproduce the r_{DM} vs. \mathcal{L} scaling of Persic et al (1996).

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